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Existence and Uniqueness of Solution for Second Order Nonlinear Integrodifferential Equations of Mixed Type in Banach Spaces*

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Abstract: In this paper, by use of a new comparison result and monotone iterative method, the unique solution and successive approxmation of the two-point boundary value problems for nonlinear integrodifferential equations of mixed type in Banach spaces are investigated. At the same time,we give out the formula in error estimate between iterative sequences and unique solution. Then apply this result to the three-point boundary value problem of three order ordinary differential equation in Banach space.

Keywords: two-point boundary value problems; monotone iterative method; unique solution; Integrodifferential equations

MSC: 34B15, 34B25.

1 Introduction

Consider the following two-point BVP for integrodifferential equations of mixed type in Banach spaces E:

$$\begin{cases} -x'' = f(t,x,x',Tx,Sx), \ \ t \in I, \\ x(0) = x'(1) = \theta. \end{cases} \tag{1.1}$$

where $f \in C[I \times E \times E \times E \times E, E], I = [0,1]$, E is a real Banach space with norm $\|\cdot\|$,θ denotes the zero element of E and

$$Tx(t) = \int_0^t k(t,s)x(s)ds, Sx(t) = \int_0^1 h(t,s)x(s)ds. \tag{1.2}$$

where $k \in C[D, R^+], h \in C[D_0, R^+], D = \{(t,s) \in R^2 : 0 \le s \le t \le 1\}, D_0 = \{(t,s) \in R^2 : 0 \le s,t \le 1\}$ and R^+ denotes the set of nonnegative real numbers. In the special case where f is uniformly continuous on $I \times B_R \times B_R \times B_R \times B_R$ for any $R > 0$ and f is increasing in x, Tx and Sx, Erbe and Guo [2] established a existence theorem on extremal solutions for BVP (1.1). In this paper, we shall use different method and somewhat different comparison results to reduce and omit these assumptions and obtain iterative sequences which converge uniformly to unique solution of BVP (1.1). The main characteristic of our conclusions is that we don't employ the compactness-type conditions and the dissipative-type conditions.

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2 Preliminaries and Lemmas

Let P be a cone in E that is a closed convex subset such that $\lambda P \subset P$ for any $\lambda \geq 0$ and $P \bigcap \{-P\} = \theta$. By means of P a partial order $\leq$ is defined as $x \leq y$ iff $y - x \in P$. A cone P is said to be normal if there exists a constant $\lambda > 0$ such that $x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq \lambda \|y\|$. The cone P is normal iff every ordered interval $[x, y] = \{z \in E : x \leq z \leq y\}$ is bounded. Let $P_c = \{u \in C[I, E] : u(t) \geq \theta, \forall t \in I\}$,where $C[I, E]$ denotes the Banach space of all continous mapping $u : I \to E$ with normal $\|u\|_c = \max_{t \in I} \|u(t)\|$. It is clear P_c is a cone of the space $C[I, E]$ and so it defines a partial ordering in $C[I, E]$. Obviously , the normality of P implies the normality of P_c and the normal conmal constants of P_c and P are the same. We shall always assume E^* is dual space of E. $P^* = \{\varphi \in E^* : \varphi(x) \geq 0, x \in P\}$ is a dual cone of P. We denote $N = \{1, 2, 3, \cdots\}, k_0 = max\{k(t, s) : (t, s) \in D\}, h_0 = max\{k(t, s) : (t, s) \in D_0\}$.

The proof of our main results in this paper will need the following Lemmas:

Lemma 2.1 ([3]) *Let $U(t, u)$ be continuous on an open $(t, u) - set$ O and $u = u_0(t)$ is the minimal solution of*

$$\begin{cases} u' = U(t, u), \\ u(t_0) = u_0. \end{cases}$$

Let $v(t)$ be a continuous differentiable function on $[t_0 - \alpha, t_0]$ satisfying the conditions

$$\begin{cases} v'(t) \leq U(t, v(t)), t_0 - \alpha \leq t \leq t_0, \\ v(t_0) \geq u_0, (t, v(t)) \in O. \end{cases}$$

Then, on a common interval of existence of $u_0(t)$ and $v(t)$,$v(t) \geq u_0(t)$.

Lemma 2.2 *Let $m(t) \in C^1[I, R]$, with $m(1) \geq 0$,satisfying*

$$m'(t) \leq M \int_0^t m(s)ds + Nm(t) + P \int_0^t k_1(t, s)m(s)ds + G \int_0^1 h_1(t, s)m(s)ds, t \in I$$

where $k_1(t, s) = \int_s^t k(t, \tau)d\tau, h_1(t, s) = \int_s^1 h(t, \tau)d\tau, M, N, P, G \geq 0$ are constants, then $m(t) \geq 0, t \in I$.

Lemma 2.3 ([4]) *Let E be a real Banach space and P be a normal cone in E ,let $D = [u_0, v_0]$ be ordered interval in $E, A : D \to E$ satisfying:*

(1)$u_0 \leq Au_0, Av_0 \leq v_0$;

(2)There exists $M \geq 0$,such that for any $u_0 \leq x \leq y \leq v_0$, we have

$$-M(y - x) \leq Ay - Ax \leq L(y - x)$$

where $L : E \to E$ is linear operator and $r(L) < 1$ ($r(L)$ is the spectral radius of L) ,then , the equation $x = Ax$ has a unique solution $x^* \in D$, and $\forall x_0 \in D$, successive iterates

$$x_n = \frac{1}{1 + M}[Ax_{n-1} + Mx_{n-1}], n = 1, 2, \cdots$$

converge to x^*,$\forall \alpha : r(L) < \alpha < 1$,there exists $n_0 \in N$,such that

$$\|x_n - x^*\| \leq 2\lambda(\frac{\alpha + M}{1 + M})^n \|u_0 - v_0\|, n \geq n_0,$$

where λ is the normal constant of P.

3 Main Results

In this section, we prove the existence and uniqueness of solution of $BVP(1.1)$ in Banach spaces. Let us list the following assumptions for convenience.

(H_1) There exist $x_0, y_0 \in C^2[I,E], x_0 \leq y_0, x_0' \leq y_0'$ such that x_0, y_0 are lower and upper solutions of $BVP(1.1$,respectively, that is

$$\begin{cases} -x_0'' \leq f(t,x_0,x_0',Tx_0,Sx_0), t \in I \\ x_0(0) \leq \theta, x_0'(1) \leq \theta, \end{cases}, \begin{cases} -y_0'' \geq f(t,y_0,y_0',Ty_0,Sy_0), t \in I \\ y_0(0) \geq \theta, y_0'(1) \geq \theta, \end{cases}$$

(H_2) Whenever $t \in I$ and $x, y \in [x_0, y_0] = \{z \in C^2[I,E] : x_0 \leq z \leq y_0\}, x \leq y, x_0' \leq x' \leq y' \leq y_0'$

$$\begin{aligned} & f(t,y,y',Ty,Sy) - f(t,x,x',Tx,Sx) \\ \geq & -M(y-x) - N(y'-x') - PT(y-x) - GS(y-x) \end{aligned}$$

where $M, N, P, G \geq 0$ are constants.

(H_3) Whenever $t \in I$ and $x_i, y_i \in E, x_i \leq y_i (i = 1,2,3,4)$,

$$\begin{aligned} & f(t,y_1,y_2,y_3,y_4) - f(t,x_1,x_2,x_3,x_4) \\ \leq & R(y_1-x_1) + E(y_2-x_2) + F(y_3-x_3) + Q(y_4-x_4) \end{aligned}$$

where $R, E, F, Q \geq 0$ are constants.

Now we prove the main results of this paper.

Theorem 3.1 *Let E be a real Banach space and P be a normal cone in E . Assume that $(H_1)-(H_3)$ hold. Then $BVP(1.1)$ has unique solution $z_* \in [x_0, y_0]$, for any $z_0 \in [x_0, y_0]$, we have $z_n(t) \rightrightarrows z_*(t), z_n'(t) \rightrightarrows z_*'(t), t \in I$, where*

$$\begin{aligned} z_n(t) = & \int_0^t \int_\tau^1 e^{N(\tau - s)}[f(s, z_{n-1}(s), z_{n-1}'(s), Tz_{n-1}(s), Sz_{n-1}(s)) \\ & - M(z_n(s) - z_{n-1}(s)) + Nz_{n-1}'(s) \\ & - PT(z_n(s) - z_{n-1}(s)) - GS(z_n(s) - z_{n-1}(s))]dsd\tau \end{aligned} \tag{3.1}$$

and $\forall \varepsilon \in (0,1)$, there exists $n_0 \in N$, such that

$$\|z_n - z_*\|_c \leq \lambda \varepsilon^n \|x_0' - y_0'\|_c, n \in N, n \geq n_0 \tag{3.2}$$

where λ is the normal constant of P.

Proof. By [2] Lemma 1, $BVP(1.1)$ is equivalent to the following IVP

$$\begin{cases} -u' = f(t, T_0u, u, T_1u, S_1u), \\ u(1) = \theta. \end{cases} \tag{3.3}$$

where

$$T_0u(t) = \int_0^t u(s)ds, T_u(t) = \int_0^t k_1(t,s)u(s)ds, S_1u(t) = \int_0^1 h_1(t,s)u(s)ds.$$

Let $u_0(t) = x_0'(t), v_0(t) = y_0'(t), t \in I$. Then $u_0(t) \leq v_0(t), t \in I$. By (H_1), it is easy to show

$$\begin{cases} -u_0'(t) \leq f(t, T_0u_0(t), u_0(t), T_1u_0(t), S_1u_0(t)), \\ u_0(1) \leq \theta. \end{cases} \tag{3.4}$$

$$\begin{cases} -v_0'(t) \geq f(t, T_0v_0(t), v_0(t), T_1v_0(t), S_1v_0(t)), \\ v_0(1) \geq \theta. \end{cases} \tag{3.5}$$

Now ,for any $\eta \in [u_0, v_0] = \{\eta \in C^1[I, E] : u_0(t) \le \eta(t) \le v_0(t), t \in I\}$, consider the linear IVP:

$$\begin{cases} -u'(t) = g(t) - MT_0u(t) - Nu(t) - PT_1u(t) - GS_1u(t), t \in I, \\ u(1) = \theta. \end{cases} \tag{3.6}$$

where

$$\begin{aligned} g(t) =& f(t, T_0\eta(t), \eta(t), T_1\eta(t), S_1\eta(t)) + MT_0\eta(t) \\ &+ N\eta(t) + PT_1\eta(t) + GS_1\eta(t). \end{aligned}$$

It is easy to see that $u \in C^1[I, E]$ is a solution of $IVP(3.6)$ iff $u \in C[I, E]$ is a fixed point of the following operator equation

$$Bu(t) = \int_t^1 e^{N(t-s)}[g(s) - MT_0u(s) - Nu(s) - PT_1u(s) - GS_1u(s)]ds$$

From the contraction mapping theorem ,the operator B has a unique fixed point $u_\eta \in C[I, E]$, that is $u_\eta \in C^1[I, E]$ is the unique solution of the $IVP(3.6)$ .Now, we define the operator A by

$$A\eta = u_\eta$$

where u_η is the unique solution of $IVP(3.6)$ with respect to η and satisfies

$$\begin{cases} -u'_\eta(t) = g(t) - MT_0u_\eta(t) - Nu_\eta(t) - PT_1u_\eta(t) - GS_1u_\eta(t), t \in I, \\ u_\eta(1) = \theta. \end{cases}$$

Evidently, $A : [u_0, v_0] \to C^1[I, E] \subset C[I, E]$, It is easy that

(1)$u_0 \le Au_0, Av_0 \le v_0$;

(2)A is monotone increasing on the order $[u_0, v_0]$;

(3)$A\eta_2 - A\eta_1 \le L(\eta_2 - \eta_1), \forall \eta_1, \eta_2 \in [u_0, v_0], \eta_1\eta_2$, where

$$L\eta = (E + N + Qh_0 + Gh_0)\int_t^1 \eta(s)ds + (R + M + Fh_0 + Pk_0)\int_t^1 (\int_0^s \eta(\tau)d\tau)ds \tag{3.7}$$

Then, we prove $r(L) = 0, \forall t \in I$, by (3.7), we have

$$\|L\eta(t)\| \le (E + N + R + M + Qh_0 + Gh_0 + Fk_0 + Pk_0)\|\eta\|_c(1 - t).$$

$$\|L^2\eta(t)\| \le (E + N + R + M + Qh_0 + Gh_0 + Fk_0 + Pk_0)^2\|\eta\|_c\frac{(1 - t)^2}{2!}.$$

In the same way,we can obtain

$$\|L^n\eta(t)\| \le (E + N + R + M + Qh_0 + Gh_0 + Fk_0 + Pk_0)^n\|\eta\|_c\frac{(1 - t)^n}{n!}, t \in I, n \in N.$$

Thus

$$\|L^n\eta\|_c = \max_{t\in I}\|L^n\eta(t)\| \le \frac{(E + N + R + M + Qh_0 + Gh_0 + Fk_0 + Pk_0)^n}{n!}\|\eta\|_c$$

So

$$\|L^n\| \le \frac{(E + N + R + M + Qh_0 + Gh_0 + Fk_0 + Pk_0)^n}{n!}.$$

Hence

$$r(L) = \lim_{n\to\infty}\|L^n\|^{\frac{1}{n}} = 0$$

It follows from the normality of P that P_c is normal cone. Hence by Lemma 2.3 $IVP(3.3)$ has unique solution $w^* \in [u_0, v_0]$, moreover, $\forall w_0 \in [u_0, v_0]$,we have $w_n(t) \rightrightarrows w^*(t), t \in I$, where

$$\begin{aligned} w_n(t) = \int_t^1 & e^{N(t-s)}[f(s, T_0w_{n-1}(s), w_{n-1}(s), T_1w_{n-1}(s), S_1w_{n-1}(s)) \\ &+ Nw_{n-1}(s) - MT_0(w_n(s) - w_{n-1}(s)) \\ &- PT_1(w_n(s) - w_{n-1}(s)) - GS_1(w_n(s) - w_{n-1}(s))]ds, n \in N \end{aligned} \tag{3.8}$$

and $\forall\varepsilon \in (0,1)$, there exists $n_0 \in N$, such that

$$\|w_n - w^*\|_c \le \lambda\varepsilon^n \|u_0 - v_0\|_c, n > n_0, n \in N.$$

Let $z_n(t) = \int_0^t w_n(s)ds$, then by (3.8),(3.1) hold and we have

$$z_n(t) = \int_0^t w_n(s)ds \rightrightarrows \int_0^t w^*(s)ds = z_*(t), t \in I,$$

$$z_n'(t) = w_n(t) \rightrightarrows w^*(t) = z_*(t), t \in I.$$

On the other hand, by $z_n(t) = \int_0^t w_n(s)ds$, we have

$$\begin{aligned}\|z_n - z_*\|_c &= \|T_0 w_n - T_0 w^*\|_c = \max_{t\in I} \| \int_0^t w_n(s)ds - \int_0^t w^*(s)ds\| \\ &\le \max_{t\in I} \|w_n(t) - w^*(t)\| = \|w_n - w^*\|_c \\ &\le \lambda\varepsilon^n \|u_0 - v_0\|_c = \lambda\varepsilon^n \|x_0' - y_0'\|_c, n \in N, t \in I.\end{aligned}$$

hence (3.2) hold. ■

4 An Example

One of the ideas in the study of high order boundary value problems for differential equations is to reduce them to lower order boundary value problems of integrodifferential equations and then employ standard techniques. We shall now show that as an application of our results we can obtain unique solution for a third order mixed boundary value problem.Consider

$$\begin{cases} -x''' = f(t, x, x', x''), \;\; t \in I, \\ x(0) = x'(0) = x''(1) = \theta. \end{cases} \tag{4.1}$$

where $f \in C[I \times E \times E \times E, E], I = [0,1]$.

Conclusion: Let E be a real Banach space and P be a normal cone in E. Suppose that

(1) There exist $x_0, y_0 \in C^3[I,E], x_0 \le y_0, x_0' \le y_0', x_0'' \le y_0''$ such that x_0, y_0 are lower and upper solutions of $BVP(4.1$,respectively, that is

$$\begin{cases} -x''' \le f(t, x_0, x_0', x_0''), t \in I \\ x_0(0) \le \theta, x_0'(0) \le \theta, x''(1) \le \theta \end{cases}, \begin{cases} -y''' \ge f(t, y_0, y_0', y_0''), t \in I \\ y_0(0) \ge \theta, y_0'(0) \ge \theta, y''(1) \ge \theta \end{cases}$$

(2) Whenever $t \in I$ and $x_i, y_i \in E, x_i \le y_i (i = 1,2,3)$,

$$f(t, y_1, y_2, y_3) - f(t, x_1, x_2, x_3) \ge -M(y_1 - x_1) - N(y_2 - x_2) - P(y_3 - x_3)$$

where $M, N, P \ge 0$ are constants.

(3)Whenever $t \in I$ and $x_i, y_i \in E, x_i \le y_i (i = 1,2,3)$,

$$f(t, y_1, y_2, y_3) - f(t, x_1, x_2, x_3) \le R(y_1 - x_1) + E(y_2 - x_2) + F(y_3 - x_3)$$

where $R, E, F \ge 0$ are constants.

Then the transformation $u = x'$ reduces (4.1) to

$$\begin{cases} -u'' = f(t, T_0 u, u, u'), \\ u(0) = u'(1) = \theta. \end{cases}$$

and consequently we obtain from theorem 3.1 that $BVP(4.1)$ has unique solution $x_* \in [x_0, y_0]$,for any $z_0 \in [x_0, y_0]$, we have

$$z_n(t) \rightrightarrows x_*(t), z_n'(t) \rightrightarrows x_*'(t), z_n''(t) \rightrightarrows x_*''(t), \; t \in I,$$

where

$$z_n(t) = \int_0^t \int_0^\xi \int_\tau^1 e^{P(\tau-s)}[f(s, z_{n-1}(s), z'_{n-1}(s), z''_{n-1}(s)) - M(z_n(s) - z_{n-1}(s)) - N(z'_n(s) - z'_{n-1}(s)) + Pz''_{n-1}(s)]dsd\tau d\xi$$

and $\forall \varepsilon \in (0, 1)$, there exists $n_0 \in N$,such that

$$\|x_n - x_*\|_c \leq \lambda\varepsilon^n \|x''_0 - y''_0\|_c, n > n_0, n \in N.$$

References

[1] K.Deimling: Nonlinear Functional Analysis. Springer-verlay, New York Berlin. (1985).

[2] L.H.Erbe, D.J.Guo: Method of upper and lower solutions for nonlinear integro-differential equations of mixed type in Banach spaces. *Appl.Anal.* 52: 143-154 (1994).

[3] P.Hartman: Ordinary Differential Equations. Birkhauser Boston, Basel, Stuttgart. (1982).

[4] J.X.Sun, L.S.Liu: Iterative Solutions for Nonlinear Operator Equation and Applications. *Acta Mathemitica Scientia*. 13(3): 141-145 (1993) (in chinese).

[5] Fengjuan Meng, Fubao Zhang: Existence of Periodic Solutions for n-dimessions Systems of Duffing Type. *International Journal of Nonlinear Science*. 8(2):162-167 (2009).

[6] Hua Ni: International Journal of Nonlinear Science The Existence, Uniqueness and Stability of Almost Periodic Solutions for Riccati Differential Equation. *International Journal of Nonlinear Science*. 5(1): 58-64 (2008).

[7] Qixiang Dong, Zhenbin Fan, Gang Li: Existence of Solutions to Nonlocal Neutral Functional Differential and Integrodifferential Equations. *International Journal of Nonlinear Science*. 5(2):140-151 (2008).

[8] Jiangbo Zhou, Anchan Mai: A Mixed Initial Boundary Value Problem for the Degasperis-Procesi Equation on a Bounded Interval. *International Journal of Nonlinear Science*. 5(3): 255-260 (2008).